A Robust Numerical Solution to a Nonlinear Time-Fractional Generalized Distributed-Order Black-Scholes Equation

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***Abstract*: In this study, we address the generalized distributed-order time-fractional Black-Scholes equation through an implicit method. We approximate the time and spatial derivatives using finite difference techniques. Furthermore, we employ a quasi-linearization technique to remove the nonlinear term, thereby simplifying the computational process. The numerical results from our method demonstrate both its accuracy and convergence rate, establishing it as a robust approach for solving financial models. This research highlights the potential of finite differences and quasi-linearization in tackling mathematical equations in financial engineering.**

***Keywords*: Generalized distributed-order; Fractional derivatives; Black-Scholes model.**

1. Introduction

In the realm of finance, an option is a crucial and widely utilized financial derivative that provides the holder with the right, but not the obligation, to buy (call option) or sell (put option) an asset at a predetermined price K (referred to as the strike price) within a specified timeframe. The practice of options trading dates back to the late 18th century in both American and European markets. However, it was not until 1973, with the establishment of the Chicago Board Options Exchange and the introduction of standardized options contracts, that this financial instrument experienced significant advancements. Consequently, determining the appropriate pricing for an option became a major challenge. In the 1970s, Black and Scholes [1] and Merton [2] developed a pioneering option-pricing model that describes the dynamic behavior of option prices over time. The Black-Scholes equation is a widely used mathematical model for valuing option prices and has been the subject of extensive research (see [3] and references therein). Numerous numerical studies have been conducted on the Black-Scholes equations, including those that incorporate jumps or stochastic volatility. Fractional Black-Scholes models have garnered increasing attention due to significant contributions by researchers such as Wys [4] and Cartea et al. [5]. These models assume that the dynamics of equity prices follow jump-diffusion processes or infinite activity Lvy processes, leading to financial derivative price dynamics that satisfy fractional partial differential equations (PDEs). The nonlinear Black-Scholes equation represents an important extension of the classical Black-Scholes model, which is foundational in financial mathematics for option pricing. Unlike the original linear model, the nonlinear version incorporates more realistic market conditions, such as transaction costs, uncertain volatility, and large investor effects, making it a more accurate tool for financial analysis. The classical Black-Scholes equation, introduced by Fischer Black and Myron Scholes in 1973, assumes constant volatility and no transaction costs, leading to a linear partial differential equation (PDE). However, these assumptions often do not hold in real markets. To address this, Corresponding author.

Researchers have developed nonlinear modifications of the Black-Scholes equation. For instance, Qiu and Lorenz (2009) studied a nonlinear Black-Scholes equation where the volatility is a function of the option's value and its second derivative, leading to a PDE with nonlinear dependence on the highest derivative [6]. This model accounts for uncertain volatility, providing a more robust framework for option pricing under varying market conditions. Another significant contribution is by Ankudinova and Ehrhardt (2007), who explored several nonlinear Black-Scholes models incorporating factors such as transaction costs and risks from unprotected portfolios [7]. Their work includes numerical solutions for European and American options, transforming the problem into a convection-diffusion equation with a nonlinear term for European options and a fully nonlinear nonlocal parabolic equation for American options. These advancements highlight the importance of nonlinear Black-Scholes equations in capturing the complexities of financial markets, offering more precise and adaptable tools for option pricing and risk management. A type of linear Black-Scholes equation with a fractional distributed derivative term was proposed and solved numerically by Zhang et al. in 2022 [8]. In this paper, we explore a novel nonlinear generalized distributed-order time-fractional Black-Scholes model as follows

with the initial condition

$$\frac{\partial \chi(x,t)}{\partial t} + \chi(x,t)\frac{{}^{\varpi}\chi(x,t)}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \chi(x,t)}{\partial x^2}$$

$$\chi(x,0) = v_0(x), \zeta_1 \le x \le \zeta_2, \qquad (2)$$

and homogeneous Dirichlet boundary conditions

$$\chi(\zeta_1,t)=\chi(\zeta_2,t)=0, 0<t\leq T. \quad (3)$$

Here $r\geq 0$ is the risk-free rate, $\sigma\geq 0$ is the volatility, $\phi\in C[\zeta_1,\zeta_2]\times C[0,T]$ and $v_0\in C^2[\zeta_1,\zeta_2]$ are considered as known functions. Here, ${}^{\infty}\partial^*/\partial t$ denotes the generalized distributed-order differential operator which is defined as follows

$$\frac{{}^{\varpi}\partial^*\chi(x,t)}{\partial t}=\int_{\alpha_l}^{\alpha_r}\varpi(\alpha,x,t)\partial_t^{\alpha}\chi(x,t)d\alpha, \quad (4)$$

where $\varpi(\alpha,x,t)$ is known density function and

$$\partial_t^{\alpha}\chi(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{1}{(t-s)^{\alpha}}\frac{\partial\chi(x,s)}{\partial s}ds$$

It is a fractional derivative of Caputo sense.

The fractional distribution derivative offers significant advantages in various fields due to its enhanced modeling capabilities, flexibility, and ability to accurately represent memory and hereditary properties. It provides a better fit for anomalous diffusion processes, which is particularly beneficial in finance for pricing derivatives and risk assessment. Additionally, its application in machine learning has improved prediction accuracy by capturing long-range dependencies and complex patterns in data. These benefits make the fractional distribution derivative a powerful tool in both theoretical research and practical applications [9]. These advantages make the fractional distribution derivative a powerful tool in both theoretical research and practical applications. The generalized fractional distributed-order Black-Scholes model has garnered significant interest recently due to its ability to generalize previous models [10], [11], [12], [13], [14], [15]. Despite its potential, numerical solutions for this model have been scarce.

The structure of the paper is as follows: In the second section, we introduce a discrete scheme for solving equation (1) using the finite difference method. The section 3 is dedicated to the error analysis of the proposed method, ensuring its reliability and accuracy. We present two numerical examples, with results illustrated through figures and tables to demonstrate the method's effectiveness in section 4. Finally, the paper concludes with a summary of findings and potential future research directions in Section 5.

2. Numerical Method

Let $x_l=\zeta_1+l\Delta x, l=0,1,\dots,L, \Delta x=\frac{\zeta_2-\zeta_1}{L}, t_n=n\Delta t, n=0,1,\dots,N, \Delta t=\frac{T}{N}, t_{n+\theta}=(n+\theta)\Delta t=t_n+\theta\Delta t \qquad t_n+\theta\Delta t$ and $\alpha_k=\alpha_l+k\Delta\alpha, k=0,1,\dots,K, \Delta\alpha=\frac{\alpha_r-\alpha_l}{K}$. Also, assume that $\varpi_k^n=\varpi(\alpha_k,x,t_n)$ and $\chi^n=\chi(x,t^n)$. Using the trapezoidal integration method, we have

$$\varpi_t^*\chi^n=\frac{\Delta\alpha}{2}\left(\varpi_0^n\partial_t^{\alpha_0}\chi^n+2\sum_{k=1}^{K-1}\varpi_k^n\partial_t^{\alpha_k}\chi^n+\varpi_K^n\partial_t^{\alpha_K}\chi^n\right)+O((\Delta\alpha)^2) \quad (5)$$

For an approximation of the fractional derivatives, we use the following L1 method

$$\begin{aligned}\partial_t^{\alpha}\chi^n&=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}(t_n-s)^{-\alpha}\frac{\partial\chi(x,s)}{\partial s}ds\\&=\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{-\alpha}\frac{\partial\chi(x,s)}{\partial s}ds\\&=\frac{1}{\Delta t\Gamma(1-\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{-\alpha}\left(\chi^{i+1}-\chi^i+O((\Delta t)^2)\right)ds\end{aligned}$$

where $J_{i,n}^{\alpha}=\int_{t_i}^{t_{i+1}}(t_n-s)^{-\alpha}ds$. By substituting (6) in (5) we obtain

$$\varpi\partial_t^*\chi^n=\frac{\Delta\alpha}{2\Delta t}\sum_{i=0}^{n-1}(\chi^{i+1}-\chi^i)\xi_i^n+O((\Delta\alpha)^2+(\Delta\alpha)(\Delta t))(7)$$

where

$$\xi_i^n(x)=\varrho_{i,0}^n(x)+2\sum_{k=1}^{K-1}\varrho_{i,k}^n(x)+\varrho_{i,K}^n(x), \varrho_{i,k}^n(x)=\frac{\varpi_k^n(x)J_{i,n}^{\alpha_k}}{\Gamma(1-\alpha_k)}$$

By putting $t=t_{n+1/2}$ in (1) and using the Crank-Nicolson method we get

$$\chi^{n+1}-\chi^n+\frac{\Delta t}{2}(\chi^{n+1\varpi}\partial_t^*\chi^{n+1}+\chi^{n\varpi}\partial_t^*\chi^n)-\frac{a\Delta t}{2}(\chi_{xx}^{n+1}+\chi_{xx}^n)+\frac{b\Delta t}{2}(\chi_x^{n+1}+\chi_x^n)$$

To get rid of the nonlinear term, we use the quasi-linearization in [16] as

$$\varpi\partial_t^*\chi^{n+1}\chi^{n+1}=\chi^{n+1}\varpi\partial_t^*\chi^n+\chi^n\varpi\partial_t^*\chi^{n+1}-\chi^n\varpi\partial_t^*\chi^n+O(\Delta t)$$

In this case, we will have

$$\chi^{n+1}-\chi^n+\frac{\Delta t}{2}(\chi^{n\varpi}\partial_t^*\chi^{n+1}+\chi^{n+1\varpi}\partial_t^*\chi^n)-\frac{a\Delta t}{2}(\chi_{xx}^{n+1}+\chi_{xx}^n)+\frac{b\Delta t}{2}(\chi_x^{n+1}+\chi_x^n)$$

Putting (7) in (9) gives the result

$$\begin{aligned}&\left(1+\frac{\Delta\alpha}{4}\left(\xi_n^{n+1}\chi^n+\sum_{i=0}^{n-1}(\chi^{i+1}-\chi^i)\xi_i^n\right)\right)\chi^{n+1}-\frac{a\Delta t}{2}\chi_{xx}^{n+1}+\frac{b\Delta t}{2}\chi_x^{n+1}\\&=\left(1+\frac{\Delta\alpha}{4}\left(\xi_n^{n+1}\chi^n-\sum_{i=0}^{n-1}(\chi^{i+1}-\chi^i)\xi_i^{n+1}\right)\right)\chi^n+\frac{a\Delta t}{2}\chi_{xx}^n-\frac{b\Delta t}{2}\chi_x^n-c\Delta t+\Delta t\phi\left(x,t_{n+\frac{1}{2}}\right)\end{aligned}$$

Where $a=\frac{\sigma^2}{2}, b=a-r, c=r$.For spatial approximation, we use central finite difference approximations

$$\chi_{xx}^n=\frac{\chi_{l+1}^n-2\chi_l^n+\chi_{l-1}^n}{(\Delta x)^2}+O((\Delta x)^2), \chi_x^n=\frac{\chi_{l+1}^n-\chi_{l-1}^n}{2\Delta x}+O((\Delta x)^2) \quad (11)$$

where $\chi_l^n=\chi(x_l,t_n)$. By substituting (11) in (10) we have the following numerical scheme

$$-c_1\chi_{l-1}^{n+1}+\Pi_l^n\chi_l^{n+1}-c_2\chi_{l+1}^{n+1}=R_l^n+O((\Delta\alpha)^2(\Delta t)+(\Delta\alpha)(\Delta t)^2+(\Delta t)^2+(\Delta x)(\Delta t))(12)$$

Where

$$\Pi_l^n=1+\frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)\chi_l^n+\sum_{i=0}^{n-1}(\chi_l^{i+1}-\chi_l^i)\xi_i^n(x_l)\right)+\frac{a\Delta t}{(\Delta x)^2}$$

$$R_l^n=c_1\chi_{l-1}^n+\Xi_l^n\chi_l^n+c_2\chi_{l+1}^n-c\Delta t+\Delta t\phi\left(x,t_{n+\frac{1}{2}}\right)$$

$$\Xi_l^n=1+\frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)\chi_l^n-\sum_{i=0}^{n-1}(\chi_l^{i+1}-\chi_l^i)\xi_i^{n+1}(x_l)\right)-\frac{a\Delta t}{(\Delta x)^2}$$

$$c_1=\frac{\Delta t}{2\Delta x}\left(\frac{a}{\Delta x}+\frac{b}{2}\right), c_2=\frac{\Delta t}{2\Delta x}\left(\frac{a}{\Delta x}-\frac{b}{2}\right), l$$

Therefore, the numerical scheme for solution problem (1)-(3) can be in the following form

$$-c_1U_{l-1}^{n+1}+P_{1,l}^nU_l^{n+1}-c_2U_{l+1}^{n+1}=Rhs_l^n \quad (13)$$

Where

$$P_{1,l}^n=1+\frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)U_l^n+\sum_{i=0}^{n-1}(U_l^{i+1}-U_l^i)\xi_i^n(x_l)\right)+\frac{a\Delta t}{(\Delta x)^2}$$

$$Rhs_l^n=c_1U_{l-1}^n+P_{2,l}^nU_l^n+c_2U_{l+1}^n-c\Delta t+\Delta t\phi\left(x,t_{n+\frac{1}{2}}\right)$$

$$P_{2,l}^n=1+\frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)U_l^n-\sum_{i=0}^{n-1}(U_l^{i+1}-U_l^i)\xi_i^{n+1}(x_l)\right)-\frac{a\Delta t}{(\Delta x)^2}.$$

Based on scheme (13) we obtain the following system of linear equations

Table 1
The maximum-norm errors and convergence rates for different values of L,N, and K in example 1

L	N,K	ε_∞^L	ρ^x	$\|$	N	L,K	ε_∞^N
4		$6.900628e-03$	-	2		$3.035511e-03$	-
8		$1.808096e-03$	1.932	4		$9.166809e-04$	1.727
16	$N=20$	$4.673662e-04$	1.952	8	$L=50$	$2.950106e-04$	1.635
32	$K=20$	$1.149771e-04$	2.023	16	$K=20$	$1.068583e-04$	1.465
64		$2.755720e-05$	2.060	32		$4.083690e-05$	1.388
128		$9.867297e-06$	1.482	64		$6.297917e-05$	0.625

$$\begin{pmatrix} P_{1,1}^n & -c_2 & & & \\ -c_1 & P_{1,2}^n & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_1 & P_{1,L-2}^n & -c_2 \\ & & & -c_1 & P_{1,L-1}^n \end{pmatrix}\begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{L-2}^{n+1} \\ U_{L-1}^{n+1} \end{pmatrix} = \begin{pmatrix} Rhs_1^n \\ Rhs_2^n \\ \vdots \\ Rhs_{L-2}^n \\ Rhs_{L-1}^n \end{pmatrix} \quad (14)$$

It is noticeable that $U_l^0, l = 1,2,\dots,L-1$ can be obtained from an initial condition (2) then $U_l^n, l = 1,2,\dots,L-1$ for $n > 0$ can be obtained using the solution of system (14).

3. Error Analysis

Let us assume that $E_l^n = \chi_l^n - U_l^n$ is the absolute error resulting from the difference between the actual value of problem (1)-(3) and the value estimated from the presented method in point (x_l, t_n).

Theorem 1. Assume that $\Delta t < (\Delta x)^2$ then if Δt tends to 0 and also $\Delta\alpha$ tends to 0 then $|E_l^n| \to 0$.

Proof. By subtracting equation (12) from equation (13), we will have

$$\begin{aligned} & -c_1 E_{l-1}^{n+1} - c_2 E_{l+1}^{n+1} \\ & + \left(1 + \frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)\chi_l^n + \sum_{i=0}^{n-1}\left(\chi_l^{i+1} - \chi_l^i\right)\xi_i^n(x_l)\right) + \frac{a\Delta t}{(\Delta x)^2}\right)\chi_l^{n+1} \\ & - \left(1 + \frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)U_l^n + \sum_{i=0}^{n-1}\left(U_l^{i+1} - U_l^i\right)\xi_i^n(x_l)\right) + \frac{a\Delta t}{(\Delta x)^2}\right)U_l^{n+1} \\ & = \left(1 + \frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)\chi_l^n - \sum_{i=0}^{n-1}\left(\chi_l^{i+1} - \chi_l^i\right)\xi_i^{n+1}(x_l)\right) - \frac{a\Delta t}{(\Delta x)^2}\right)\chi_l^n \\ & - \left(1 + \frac{\Delta\alpha}{4}\left(\xi_n^{n+1}(x_l)U_l^n - \sum_{i=0}^{n-1}\left(U_l^{i+1} - U_l^i\right)\xi_i^{n+1}(x_l)\right) - \frac{a\Delta t}{(\Delta x)^2}\right)U_l^n \end{aligned}$$

By putting $n = 0$ in (15) we obtain

$$\begin{aligned} & -c_1 E_{l-1}^1 - c_2 E_{l+1}^1 + \left(1 + \frac{\Delta\alpha}{4}\xi_0^1(x_l)\chi_l^0 + \frac{a\Delta t}{(\Delta x)^2}\right)\chi_l^1 - \left(1 + \frac{\Delta\alpha}{4}\xi_0^1(x_l)U_l^0 + \frac{a\Delta t}{(\Delta x)^2}\right)U_l^1 \\ & = \left(1 + \frac{\Delta\alpha}{4}\xi_0^1(x_l)\chi_l^0 - \frac{a\Delta t}{(\Delta x)^2}\right)\chi_l^0 - \left(1 + \frac{\Delta\alpha}{4}\xi_0^1(x_l)U_l^0 - \frac{a\Delta t}{(\Delta x)^2}\right)U_l^0 \end{aligned}$$

Since $\chi_l^0 = U_l^0$ and $\Delta t \to 0$ then $c_1 \to 0$ and $c_2 \to 0$ and so from (16) we get

$$\lim_{\Delta t \to 0}\left(1 + \frac{\Delta\alpha}{4}\left|\xi_0^1(x_l)\chi_l^0\right|\right)\left|E_l^1\right| = 0 \quad (17)$$

Whereas $\Delta\alpha \to 0$ therefore

$$\left|E_l^1\right| \to 0 \quad (18)$$

According to equation (15) and assumptions, we have

$$\left|E_l^{n+1}\right| \le \left|E_l^n\right| \quad (19)$$

Equation (19) along with (18) gives what we wanted.

4. Numerical Results

In this section, we study the numerical results of the implementation of the presented method in two examples. The versatility and the accuracy of the methods are measured using the maximum absolute error norms $\varepsilon_\infty^L = \max_{0\le l\le L}\left|\chi(x_l, t_N) - U(x_l, t_N)\right|, \varepsilon_\infty^N = \max_{0\le n\le N}\left|\chi\left(x_{[L/2]}, t_n\right) - U\left(x_{[L/2]}, t_n\right)\right|$, and the convergence rates are defined as follows

$$\rho^x = \frac{\log\left(\frac{\varepsilon_\infty^L}{\varepsilon_\infty^{2L}}\right)}{\log(2)}, \rho^t = \frac{\log\left(\frac{\varepsilon_\infty^N}{\varepsilon_\infty^{2N}}\right)}{\log(2)}.$$

Example 1. Let $T = 1, \zeta_1 = 0, \zeta_2 = 1, \alpha_l = 0.2, \alpha_r = 0.8, r = 0.02, \sigma = 0.1, v_0(x) = \sin(\pi x)$ and $\varpi(x,t,\alpha) = 0.01\Gamma(1-\alpha)\sqrt{(x+1)(t+1)}$. The exact solution is assumed to be $\chi(x,t) = (t+1)\sin(\pi x)$ and $\phi(x,t)$ can be obtained from exact solution as follows

$$\begin{aligned} \phi(x,t) = & c + \sin(\pi x) + b\pi\cos(\pi x)(t+1) \\ & - \left(\sin^2(\pi x)\sqrt{(t+1)(x+1)}(t+1)\left(\mathrm{logint}\left(t^{1/5}\right) - \mathrm{logint}\left(t^{4/5}\right)\right)\right)/100 + a\pi^2\sin(\pi x)(t+1) \end{aligned}$$

in MATLAB syntax. The norm of errors and the order of convergences for different values of L, N, and K are reported in Table 1. Also, the numerical solution and its related absolute errors are shown in Figure 1.

Example 2. Let $T = 1, \beta_1 = 0, \beta_2 = 1, \beta_* = 0.2, \beta^* = 0.6, r = 0.02, \sigma = 0.1, v_0(x) = \sin(\pi x)$, $\gamma(x,t,\beta) = 0.01\Gamma(1-\beta)\sqrt{(x+1)(t+1)}$. The exact solution is assumed to be $v(x,t) = (1+t)\sin(\pi x)$ and $f(x,t)$ can be obtained from exact solution as follows

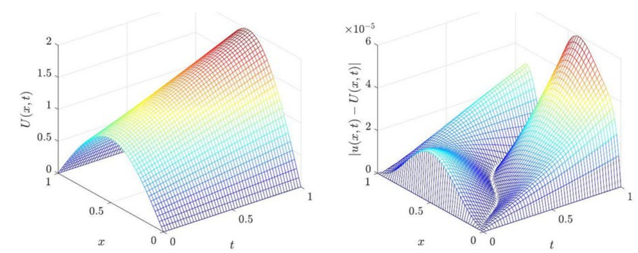

Fig. 1. Numerical solution (left) and the absolute errors for all mesh points in example 1 with $N = 50$, $L = 50$ and $K = 20$

$$\begin{aligned} f(x,t) = & c + \sin(\pi x) - 0.01\left(\sin(\pi x)((t+1)(x+1))^{1/2}\left(\mathrm{logint}\left(t^{2/5}\right) - \mathrm{logint}\left(t^{4/5}\right)\right)\right) \\ & + b\pi\cos(\pi x)(t+1) + a\pi^2\sin(\pi x)(t+1) \end{aligned}$$

In MATLAB syntax. The numerical and exact solutions for $N = 40, L = 40$, and $K = 30$ are shown in Figure 2. In addition, the absolute errors in all mesh points are shown in Figure 3 for $N = 40, L = 40$, and $K = 30$.

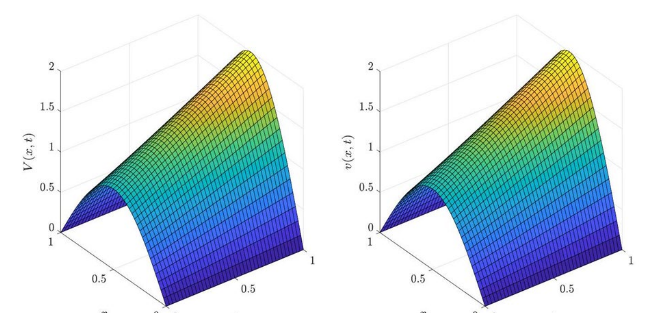

Fig. 2. Numerical solution (left) and the exact solution (right) for all mesh points in example 2 with $N = 40, L = 40$, and $K = 30$

5. Conclusion

In conclusion, this study effectively addresses the generalized distributed-order time-fractional Black-Scholes equation using an implicit method. By approximating the time and spatial derivatives with finite difference techniques and employing quasi-linearization to eliminate the nonlinear term, we have simplified the computational process. The numerical results confirm the accuracy and convergence rate of our method, establishing it as a robust approach for solving financial models. This research underscores the potential of finite differences and quasi-linearization in addressing complex mathematical equations in financial engineering.

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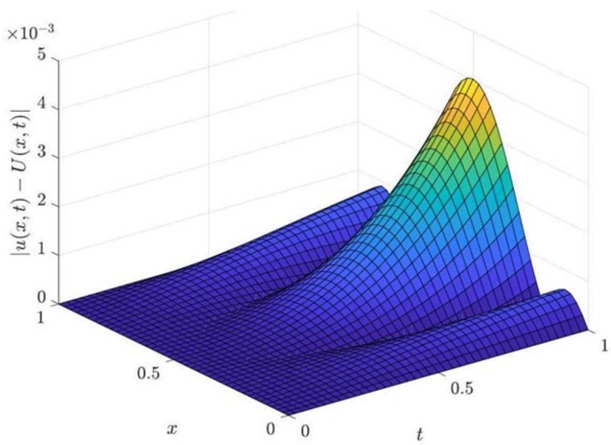

Fig. 3. The absolute errors for all mesh points in example 2 with $N = 40, L = 40$ and $K = 30$

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