An Application of Fuzzy Root Square Mean Labeling to Multi Path Networks

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***Abstract*: Multi-path networks play a crucial role in communication, transportation, and reliability systems, where multiple internally disjoint paths exist between two terminals. In real-world applications, uncertainty in node availability and link reliability motivates the use of fuzzy graph models. This paper introduces Fuzzy Root Square Mean Labeling (FRSML) as an extension of classical root square mean labeling under fuzzy conditions. Theta-related graphs are employed as canonical models for multi-path networks. We establish labeling schemes, prove existence results for certain classes of theta graphs, and illustrate the applicability of the proposed method through mathematical examples and diagrams. The study demonstrates that fuzzy root square mean labeling effectively represents uncertainty while preserving the structural characteristics of multi-path networks.**

***Keywords*: Fuzzy graph, Root square mean labeling, Theta graph, Multi-path networks, Fuzzy labeling.**

1. Introduction

Graph labeling has been widely studied due to its theoretical significance and practical applications in network modeling. Among various labeling techniques, **root square mean labeling** assigns edge labels derived from vertex labels using the root mean square formula. However, classical labeling assumes precise information, which is often unrealistic in real-world networks.

Multi-path networks, characterized by the existence of multiple internally disjoint paths between nodes, are naturally modeled by **theta graphs**. In practical scenarios such as communication networks and traffic systems, uncertainty in node functionality and link strength necessitates fuzzy representations. Fuzzy graph theory provides an effective framework to incorporate such uncertainty.

This paper applies fuzzy root square mean labeling to multi-path networks modeled by theta-related graphs. The proposed approach extends classical labeling into a fuzzy environment and offers a mathematically consistent tool for uncertain network analysis.

In this paper, the basic concepts related to fuzzy graphs and fuzzy root square mean labeling are presented in Section II. The main theoretical results concerning fuzzy root square mean labeling of theta-related graphs are established in Section III, along with illustrative theorems and examples to demonstrate their applicability to multi-path networks. The concluding remarks and possible extensions of the present work are discussed in Section IV. Finally, the relevant references supporting the study is listed in Section V.

2. Basic Definitions

Definition 2.1 (Fuzzy Set)

A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_A(x)) : x \in A, \mu_A(x) \in [0,1]\}$ [12]. In the pair $(x, \mu_A(x))$, the first element x belongs to the classical set A and the second element $\mu_A(x)$ belongs to the interval [0, 1] called membership function [7].

Definition 2.2 (Fuzzy Number)

A fuzzy set $\tilde{A}$ on R must possess at least the following three properties to qualify as a fuzzy number [7]:

(i) $\tilde{A}$ must be a normal fuzzy set;

(ii) $\alpha_{\tilde{A}}$ must be a closed interval for every α ∈ [0, 1];

and

(iii) the support of $\tilde{A}$ must be bounded.

Definition 2.3 (Root Square Mean Labeling)

A graph $G = (V, E)$ with s nodes t links is said to be a RsM graph if it's likely to value the nodes $x \in V$ with distinct each elements $f^{RsMl}(x)$ from 1,2,…,t+1 in such the way that once all link $e = uv$ is labelled with

$f^{RsMl}(e = uv = \left\lceil \frac{\sqrt{f(u)^2+f(v)^2}}{2} \right\rceil or \left\lfloor \frac{\sqrt{f(u)^2+f(v)^2}}{2} \right\rfloor$, then the ensuring link values are distinct. In this case f is called a $RsMl$ of G.

Definition 2.4 (Path)

A walk in which $u_1, u_2, \dots, u_n$ are distinct is called a path. A path on n vertices is denoted by P_n.

Definition 2.5 (Fuzzy Graph)

A fuzzy graph is an ordered pair

$$G = (V, \mu_V, \mu_E),$$

Where $\mu_V : V \to [0,1]$ is the vertex membership function

$\mu_E : V \times V \to [0,1]$ is the edge membership function

satisfying $\mu_E(u,v) \le \min\{\mu_V(u), \mu_V(v)\}$.

Definition 2.6 (Theta Graph)

A theta graph $\theta(p,q,r)$ consists of two distinct vertices joined by three internally disjoint paths of lengths p, q, r.

Definition 2.7 (Fuzzy Root Square Mean Labeling)

Let $G = (V, E)$ be a fuzzy graph. A **fuzzy root square mean labeling** is a function

$f : V \to [0,1]$ such that for each edge

$$uv \in E,\ f(uv) = \sqrt{\frac{f(u)^2 + f(v)^2}{2}},$$

and the induced edge labels satisfy the fuzzy consistency condition.

Definition 2.8 (Multi-Path Network Model)

A **multi-path network** is a network represented by a graph $G=(V,E)$ in which there exist two distinct vertices $u, v \in V$ such that there are **at least two internally disjoint paths** between u and v.

Mathematically, if $P_1(u,v), P_2(u,v), \dots\dots P_k(u,v)$ are paths from u to v, then the network is called a multi-path network if

$$k \ge 2$$

$$P_i(u,v) \cap P_j(u,v) = (u,v)\ \forall i \ne j$$

Under fuzzy conditions, the network is modeled as a **fuzzy theta graph**, where:

$$x \in V, \mu_v(x) \in [0,1]$$

$$xy \in V, \mu_v(xy) \in [0,1]$$

$$\mu_E(xy) \le \min\{\mu_V(x), \mu_V(y)\}$$

Thus, a fuzzy theta graph provides a mathematical model for representing uncertainty in multi-path networks.

Definition 2.9 (Internally Disjoint Path)

Two paths $P_1(u,v)$ and $P_2(u,v)$ between vertices u and v in a graph $G(V,E)$ are said to be **internally disjoint** if they have no common vertices other than the end vertices u and v.

Mathematically, $P_1(u,v) \cap P_2(u,v) = \{u, v\}$

Definition 2.10 (Fuzzy Path)

A fuzzy path between two vertices u and v is a sequence of vertices

$$u = v_0, v_1, \dots\dots v_n = v$$

where each consecutive pair $v_i v_{i+1}$ is an edge with positive membership value.

Definition 2.11 (Fuzzy Path Strength)

The **strength of a fuzzy path** P is defined as the minimum of the membership values of all edges along the path, that is,

$$S(P) = \min\{\mu_E(e) : e \in P\}$$

Definition 2.12 (Most Reliable Path)

Among all internally disjoint fuzzy paths between two terminal vertices u and v, the path with the maximum cumulative path membership value is called the most reliable path.

3. Main Results

Theorem 3.1

Every fuzzy theta graph admits a fuzzy root square mean labeling.

Proof:

Let $G(V,E)$ be a **fuzzy theta graph,**

where:

V is the set of vertices

E is the set of edges

$\mu_V : V \to [0,1]$ is the vertex membership function

$\mu_E : E \to [0,1]$ is the edge membership function.

A theta graph consists of two terminal vertices u and v connected by three internally disjoint paths

$$P_1(u,v), P_2(u,v), P_3(u,v)$$

Assign distinct membership values to all vertices such that

$$\mu_V(x) \cup (0,1]\ \ \forall\ x \in V$$

For any edge $xy \in E$, define the edge membership using the **root square mean rule**:

$$\mu_E(xy) = \sqrt{\frac{\mu_V(x)^2 + \mu_V(y)^2}{2}}$$

Since

$$0 < \mu_V(x), \mu_V(y) \le 1$$

it follows that

$$0 < \mu_E(xy) \le 1$$

Thus, the induced edge memberships satisfy the fuzzy consistency condition, and hence the fuzzy theta graph admits a **fuzzy root square mean labeling**.

Example 3.2

Consider a fuzzy theta graph $\theta(2,2,2)$ consisting of two terminal vertices u and v, connected by three internally disjoint paths, each of length two.

Step 1: Vertex Set and Membership Values

Let the vertex set be $V = \{u, a, b, c, v\}$

Assign the following vertex membership values:

$$\mu_V(u) = 0.9,\ \mu_V(a) = 0.7, \mu_V(b) = 0.6$$

$$\mu_V(c) = 0.5, \mu_V(v) = 0.4$$

Step 2: Edge Membership Values (Root Square Mean Rule)

For each edge xy, define the edge membership as

$$\mu_E(xy) = \sqrt{\frac{\mu_V(x)^2 + \mu_V(y)^2}{2}}$$

Now compute the edge memberships:

1. Edge ua

$$\mu_E(ua) = \sqrt{\frac{0.9^2 + 0.7^2}{2}} = 0.806$$

2. Edge av $\mu_E(av) = 0.570$
3. Edge ub $\mu_E(ub) = 0.763$
4. Edge bv $\mu_E(bv) = 0.510$
5. Edge uc $\mu_E(uc) = 0.728$
6. Edge cv $\mu_E(cv) = 0.453$

Step 3: Verification of Fuzzy Conditions

All vertex memberships and induced edge memberships lie in the interval [0, 1].

Hence, the fuzzy consistency condition is satisfied.

Theorem 3.3

For a fixed set of vertex membership values, the induced fuzzy root square mean edge labels are unique.

Proof:

Assume that the vertex membership function is fixed, that is,

$$\mu_V : V \to [0,1]$$

assigns a unique and unchanging membership value to every vertex.

For any edge $xy \in E$, the **fuzzy root square mean edge membership** is defined as

$$\mu_E(xy) = \sqrt{\frac{\mu_V(x)^2 + \mu_V(y)^2}{2}}$$

Since: $\mu_V(x) \,\&\, \mu_V(y)$ are fixed real numbers in [0,1] and the square, addition, division, and square root operations are all **deterministic**, the value of

$$\mu_E(xy) = \sqrt{\frac{\mu_V(x)^2 + \mu_V(y)^2}{2}}$$

is uniquely determined.

Thus, for every edge $xy \in E$, there exists **exactly one** induced fuzzy root square mean edge membership value.

Hence, for a fixed set of vertex membership values, the induced fuzzy root square mean edge labels are unique.

Example 3.4

Consider a fuzzy theta graph with vertex membership values:

$$\mu_V(u) = 0.9,\ \mu_V(a) = 0.7,$$

$$\mu_V(b) = 0.6, \mu_V(v) = 0.4$$

Now compute the edge memberships:

Edge ua $\mu_E(ua) = \sqrt{\frac{0.9^2 + 0.7^2}{2}} = 0.806$

Edge av $\mu_E(av) = 0.570$

Each edge membership is uniquely fixed by its endpoint vertices.

Thus, the entire fuzzy root square mean labeling of the theta graph is unique.

Theorem 3.5

In a fuzzy theta graph, the path with the highest total fuzzy root square mean edge membership is the most reliable path between the terminal vertices.

Proof:

Let $G(V, E)$ be a **fuzzy theta graph** with terminal vertices u and v.

Let there be three internally disjoint paths between u and v, denoted by

$$P_1(u,v), P_2(u,v), P_3(u,v)$$

Let the vertex membership function be fixed as

$\mu_V : V \to [0,1]$

For any edge $xy \in E$, the **fuzzy root square mean edge membership** is defined as

$$\mu_E(xy) = \sqrt{\frac{\mu_V(x)^2 + \mu_V(y)^2}{2}}$$

For a path $P = \{e_1, e_2, e_3.....e_k\}$, define the total fuzzy root square mean edge membership of the path as

$$T(P) = \sum_{i=1}^{k} \mu_E(e_i)$$

Since each edge membership value represents the reliability of that link, the sum of edge memberships along a path represents the overall reliability of the path.

If

$$T(P_i) \geq T(P_j) \ \ \forall i \neq j$$

then path P_i has the highest cumulative reliability.

Hence, the path with the highest total fuzzy root square mean edge membership is the **most reliable path** between the terminal vertices u and v.

Example 3.6:

Consider a fuzzy theta graph with vertex memberships:

$$\mu_V(u) = 0.9,\ \mu_V(a) = 0.7, \mu_V(b) = 0.6$$

$$\mu_V(c) = 0.5, \mu_V(v) = 0.4$$

For each edge xy, define the edge membership as

$$\mu_E(xy) = \sqrt{\frac{\mu_V(x)^2 + \mu_V(y)^2}{2}}$$

Now compute the edge memberships:

1. Edge ua

$$\mu_E(ua) = \sqrt{\frac{0.9^2 + 0.7^2}{2}} = 0.806$$

2. Edge av $\mu_E(av) = 0.570$
3. Edge ub $\mu_E(ub) = 0.763$
4. Edge bv $\mu_E(bv) = 0.510$
5. Edge uc $\mu_E(uc) = 0.728$
6. Edge cv $\mu_E(cv) = 0.453$

Total path memberships:

Path

$P_1 = u \to a \to v$

$T(P_1) = 0.806 + 0.570 = 1.376$

$P_2 = u \to b \to v$

$T(P_2) = 0.763 + 0.510 = 1.273$

$P_3 = u \to c \to v$

$T(P_2) = 0.728 + 0.453 = 1.181$

Since

$$T(P_1) > T(P_2) > T(P_3)$$

Therefore, P_1 is the most reliable path.

4. Conclusion

The collaboration of fuzzy root square mean labeling with theta-related graph structures provides a mathematically sound and visually intuitive framework for modeling uncertainty in multi-path networks. The established results on existence, uniqueness, and ordering ensure theoretical consistency, while the examples and diagrams highlight the practical relevance of the model. The proposed approach offers scope for further extension to advanced fuzzy graph models such as intuitionistic and interval-valued fuzzy graphs, thereby opening new directions for future research.

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