Investigation Of Projectile Motion with Altitude Dependent Quadratic Drag Force

Kaveri Sinha Mahapatra [1], Rhombik Roy [2]

[1]Student, North Point Senior Secondary Boarding School, Kolkata 700059, India.

[2]Student, Department of Physics, Presidency University, Kolkata 700073, India.

Corresponding Author: sinhamahapatrakaveri@gmail.com

Abstract: The problem of projectile motion without frictional force is an age-old problem as it is exactly solvable. The problem has also been revisited in a resisting medium with quadratic damping. The damping constant depends on the density of the medium. All the recent calculations solve the problem with a constant dragging force. However, if the projectile is launched from a height, it will traverse several layers of air of wide varying density specially during the downward motion. As the dragging force is quadratic with the velocity, the altitude dependent dragging force will have immense effect on the trajectory if it is projected with high velocity. We apply a simple python code to study the trajectory of the projectile in a very realistic situation as varying dragging force is taken into account. We present an exhaustive study with varying launching height and launching angle. We also obtain a graphical nature of dependence of optimized angle to obtain the maximum range.

Key Words: —*Projectile motion, air drag, optimized angle.*

I. INTRODUCTION

The classic problem of a projectile motion in a constant gravitational field has a long history and is well discussed in high school physics text book. In this problem, a point mass projectile is thrown at an angle to the horizon, in absence of air drag, the motion is exactly solvable. The trajectory of the projectile becomes a parabola. So, physics students are quite familiar with the solution of projectile motion in the absence of drag. However, the projectile motion in a resisting medium has developed interest in the study of ballistic and in application of various sports [1-5].

In general, in the motion of spherical ball through a viscous fluid with low Reynolds number, the drag force is given by $F = 6\pi\eta r v$, where η is the viscosity of the fluid and v is the velocity of the ball.

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However, as the projectiles are often started off with very high speed close to 40 m/s or more, a reasonable approximation to the drag force is

$F_D = C_D \rho A v^2/2$ [6-7], Where the drag coefficient

$C_D = 0.5$ for a sphere shape. The cross-sectional area is $A=\pi r^2$.

The equations of motion and finding the trajectory under the drag force is a nontrivial problem. Although the motion of a falling raindrop under the quadratic drag force is exactly solvable, for 2-dimensional motion, no such simple solutions exist. However, one can develop approximate solution to the equations of motion or write a simple numerical code which would have a great pedagogical interest. We find a good set of references [8-12] which exhibit the renewed interest in this problem since last decade. The interesting work in this direction is the approximate analytical description of the projectile motion with a quadratic drag force [10-12]. It provides analytic formula for several parameters of projectile motion approximately. Thus, the motion of a projectile under a ***constant*** quadratic drag force is a well understood problem.

However, there are some special cases where the motion of a missile resembles projectile motion, only according to the structure, the drag coefficient and cross-sectional area will be properly taken care of.

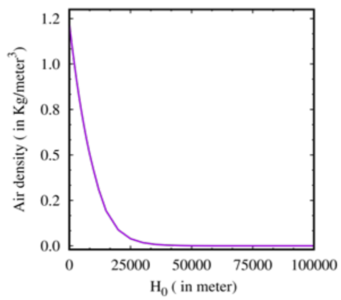

Fig.1. Variation of air density with the altitude. Data from Ref [13] are plotted. Where H_0 stands for the altitude.

The situation will be complex when some animated arms are launched to a particular height of order of few tens of km and then projected. The key difference in this case, the projectile will traverse the medium of air with varying density, *i.e.,* varying drag force. Thus, the conception of constant drag force is no longer valid. The newly defined drag force is $F_D (y)= C_D \rho(y)Av^2/2$, density of air is now function of altitude [13] and as shown in Fig.1

Thus, the motivation of the present work is to revisit the problem of projectile motion under a varying drag force. We use a python code for the entire problem. For numerical computation we consider the motion of a baseball of fixed radius and fixed mass which is projected from a height of several tens of km. Apart from taking care of the varying air density, we also take the appropriate value of acceleration due to gravity (g) when the projectile is launched from 50 km height or more. During the upward motion, as the air is already dilute, the effect of drag force is negligible. However, its downward journey is strongly affected by the high dense air. We observe strong impact of air density close to few thousands meter close to the ground. Instead of asymptotic fall the drag makes it vertically fall of the baseball. So close to the ground the curvature of the trajectory changes suddenly.

Change in the range (ΔR) due to varying drag force also strongly depends on the launching height H_0. We obtain a smooth dependence on ΔR on H_0, having a peak near 20 km launching height for particular choice of parameter. The calculation of optimized angle to obtain the maximum range is also an important parameter in the projectile motion. As our calculation consider the most realistic case of varying drag, the calculation of optimized angle would be helpful for future study of missile launched from a specific height. We observe that unlike the case of no drag where optimized angle is 45°, here we get a distribution.

However, we conclude that when the projectile is launched from a height of several tens of km and with large velocity, it should be projected almost horizontally or with angle of few degrees to get maximum range.

The paper is sectioned as follows. In Section II, we give the basic equations which are solved by the numerical code. In Section III, we present our numerical results with discussion. Section IV deals with the conclusion and open questions. References are given at the end of the manuscript.

II. Equations Of Motion

The magnitude of the drag force can be written as

$$F_D = Dv^2 \tag{1}$$

Where v is the projectile speed relative to the air. D is the coefficient which is $C_D \rho A$. During computation, we make the density ρ altitude dependent. Here we try to develop the basic equations which we use in the python code to get the trajectory. We further assume that the air is still.

Thus

$$F_D^x = -D\, v v_x \tag{2}$$

$$F_D^y = -D\, v v_y \tag{3}$$

$$v = \sqrt{(v_x^2 + v_y^2)} \tag{4}$$

From Newton's second law

$$a_x = -D/m\ v\ v_x \tag{5}$$

$$a_y = -g\ -D/m\ v\ v_y \tag{6}$$

Note that a_x and a_y are constantly changing as velocity component changes. However, choosing a very short time interval Δt, we assume that acceleration components are essentially constant. Thus during the time interval Δt, the average x-velocity (v_x) changes by $\Delta v_x = a_x\ \Delta t$ and the y-velocity (v_y) changes by $\Delta v_y = a_y\ \Delta t$.

Thus

$$v_x + \Delta v_x = v_x + a_x\ \Delta t \tag{7}$$

$$v_y + \Delta v_y = v_y + a_y \Delta t \quad (8)$$

To calculate the change in x in time Δt, we further explain the average velocity. The average x-velocity is the average of v_x and $v_x + \Delta v_x$, that is $v_x + \Delta v_x / 2$.

Thus, change in x (Δx) in time Δt is given by

$$\Delta x = v_x \Delta t + ½ a_x (\Delta t)^2 \quad (9)$$

Similarly

$$\Delta y = v_y \Delta t + ½ a_y (\Delta t)^2 \quad (10)$$

Thus, the coordinate of the projectile at any given instant of time is

$$x + \Delta x = x + v_x \Delta t + ½ a_x (\Delta t)^2 \quad (11)$$

$$y + \Delta y = y + v_y \Delta t + ½ a_y (\Delta t)^2 \quad (12)$$

Where

$$v_x = v_0 \cos\theta \quad (13)$$

$$v_y = v_0 \sin\theta \quad (14)$$

θ is the angle of projection.

We choose the initial values x, y, v_x, v_y, and t and keep the time interval of Δt = 0.001 s. For the whole computation we keep the radius r= 0.0366 m and mass m = 0.145 kg and C_D = 0.5. For each time step, the dependence of air density on the altitude is taken from the Ref [13]. According to the data points, the drag coefficient is calculated at each time step and is fed to the program.

We run the computation in a loop of maximum t, obtain the file for y(t) vs x(t) and draw the trajectory.

III. NUMERICAL RESULTS

At the very first we check for the accuracy of our python code.

We compare our numerical results with the analytical formulas of projectile parameters by Chudinov [11] for the case of constant drag. In this reference, the author considered the drag force F =mgkv^2. Thus we use the relation between k and D as k = ρ C_D A/2mg = D/mg. The comparison is presented in Table 1. The comparison is good and our python code results consistent output.

Table.1. Comparison of numerical results with analytical calculation [11] for the case of constant drag force. Chosen parameter: v_0 = 40m/s, θ = 45°, g = 9.81 m/s^2, k = 0.000625 s^2/m^2

Projectile parameter	Analytical results	Numerical results
Maximum Height (m)	30.1	29.83
Range (m)	95.7	94.98
Time of flight (s)	4.96	4.91

Next, we apply the code for the realistic case, when a projectile is launched to a height of 100 km and projected with velocity v_0 = 50m/s with an angle 0f 40 degrees. We choose this parameter to exhibit the impact of varying air density on the projectile's trajectory. The trajectory is plotted in Fig.2 for the three cases. Case 1 is the result for no drag. Case 2, corresponds to the case of constant drag.

We have utilized ρ = 5.07 X 10^(-7) kg/m^3, this is the density of air at the altitude of 100 km. Comparison to the no drag case, it is expected that the range will be shorter but the projectile has a smooth fall. Case 3 corresponds to the case where explicit dependence of air density on the altitude has been taken care of. Up to the height of 12 km from the ground the trajectory is almost indistinguishable with the zero-drag case. However, before reaching the ground, it enters a heavily dense and widely changing drag with the altitude. With velocity 50m/s and strong variation of air density, the projectile feels very strong drag force and makes a sharp fall.

Next, we calculate the change in the range ΔR as a function of different height for fixed velocity 60m/s and angle of projection 40°.

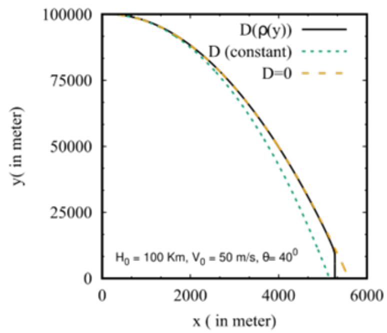

Fig.2. Trajectory of the projectile, initially launched at the height of 100 km and projected with velocity of 50m./s and with angle 40°. The three graphs correspond to the three cases: drag zero, constant drag, drag is a function of altitude.

We define ΔR as the difference between the range without drag and the range with drag. For each computation the varying drag force has been taken into account. It is plotted in Fig.3. For this particular velocity and projection angle choice we obtain the maximum change in ΔR around 20 km.

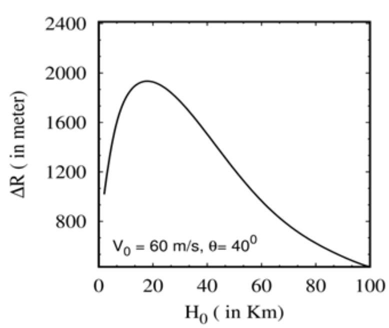

Fig.3. Plot of change in Range (ΔR), when the projectile is launched at different height before it is fired. The velocity of projection is 60m/s and angle of projection 40°. Maximum changes occur at 20 km height.

For the projectile motion it is also instructive to find the optimized angle for a fixed velocity to get the maximum range. In Fig. 4, we plot the range (in varying drag force) as a function of angle of projection in the range from 0°to 90°for the fixed velocity of projection 60m/s and for different initial height.

Each graph attains a peak at a particular angle of projection, we call it as optimized angle θ_p. With increase in launch angle the range smoothly decreases to zero.

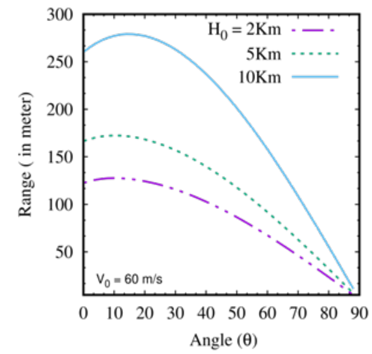

Fig.4. Variation of the range for different launch angle. The initial heights are kept to 2km, 5km, 3 km. Results for other launching heights are not plotted for clarity. The peak value corresponds to optimized angle to obtain the maximum range.

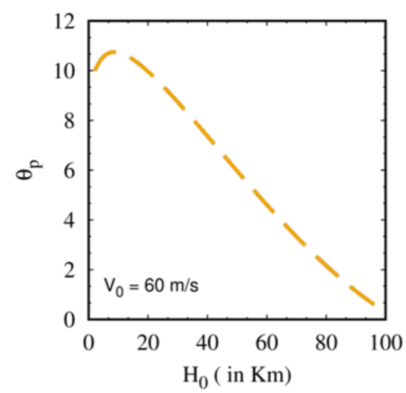

Fig.5. Variation of optimized angle as a function of launch height.

In Fig. 5 we plot the variation of optimized angle as a function of wide range of launching height. If we do not consider the effect of drag, optimized angle will go closely to zero with gradual increase in launch height. However here we get a different distribution due to the effect of varying drag. Of course, if the projectile is launched from a large height (100 km or more) to get the maximum range the projectile should be fired horizontally.

In Fig.6, we plot the same but for a fixed launching height and different velocity. With increase in velocity the optimized angle smoothly increases. Thus from Fig. 5 and Fig.6 we observe that optimized angle has a very strong dependence both on the height of launching tower as well as on the initial velocity of projection.

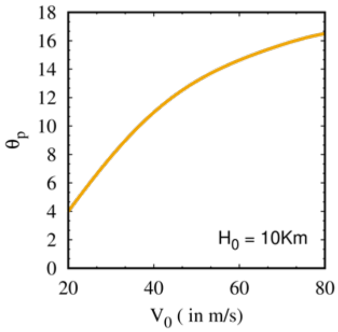

Fig.6. Variation of optimized angle as a function velocity

IV. Conclusion

In this manuscript we have considered the most generic case where the projectile will travel through the air medium of varying density before it strikes the ground. The old research works in this direction consider the throwing of a projectile from the ground where it is guaranteed that the projectile will travel through the constant dragging force. However, in many cases the projectile motion replicates the missile launching. It is quite common, especially for the animated arms that before projection they are launched at a particular height as per corresponding requirement. However under this situation the projectile will travel through the air medium whose density is widely varying; in the first ten thousands meters it is close to 1.1 kg/m^3 to 0.5 kg/m^3. After that it quickly decreases by a factor of 10 in each consecutive layer of height close to 1 to 2 km. Thus, when a projectile is projected from a launching pad of significant height it will face tremendous effect of air drag just before hitting the ground. As this a non-trivial problem we have provided a simple but self-consistent python code to tackle the realistic situation. The calculation can be easily carried out for massive projectile which may resemble the launching of missile.

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