On The Integer Solutions of The Diophantine Equations $x^2-(a^2b^2+b)y^2-(4c+2)x+4(a^2b^2+b)y-4(a^2b^2+b-c^2-c)=0$

Sriram S[1], Veeramallan P[2]

[1]*Assistant Professor, National College (Affiliated to Bharathidasan University), Tiruchirappalli, Tamilnadu, India.*

[2]*P G Assistant in Mathematics, GHSS, Yethapur, Salem, Tamilnadu, India.*

Corresponding Author: sriram.priya02@yahoo.com

Abstract: Using the theory of Pellian equations, we show that the Diophantine equations E: $x^2-(a^2b^2+b)y^2-(4c+2)x+4(a^2b^2+b)y-4(a^2b^2+b-c^2-c)=0$ have infinitely many nontrivial integer solutions (x,y) by using Generalized Biperiodic Fibonacci and lucas sequence. We also derive some recurrence relations on the integer solutions (x,y) of E. AMS Subject Classification: 11D09.

Key Words: —*Diophantine equation, Pell equation, linear transformation, continued fraction, Generalized Biperiodic Fibonacci and Lucas sequence.*

I. Introduction

A Diophantine equation is a polynomial equation $P(x_1,x_2,\ldots,x_n)=0$ where the polynomial P has integral coefficients and one is involved in options for which all the unknowns take integer values. For example, $x^2+y^2=z^2$ and $x=3$, $y=4$, $z=5$ is one of its infinitely many solutions. Another instance is $x+y=1$ and all its options are given with the aid of $x=t, y=1-t$ the place t passes through all integers. A third instance is $x^2+4y=3$. This Diophantine equation has no solutions, though be aware that $x=0, y=\frac{3}{4}$ is an answer with rational values for the unknowns. Diophantine equations are wealthy in variety. Two – variable Diophantine equation have been a situation to large research, and their concept constitutes one of the most beautiful, most complex section of mathematics, which although nonetheless maintains some of its secrets and techniques for the subsequent generation of researchers.

Manuscript revised December 08, 2022; accepted December 09, 2022. Date of publication December 10, 2022.

This paper available online at www.ijprse.com

ISSN (Online): 2582-7898; SJIF: 5.59

In this paper, we look into superb essential options of the Diophantine equation $\boldsymbol{x^2-(a^2b^2+b)y^2-(4c+2)x+4(a^2b^2+b)y-4(a^2b^2+b-c^2-c)=0}$ which is modified into a Pell's equation and is solved with the aid of Generalized Biperiodic Fibonacci and lucas sequence.

II. The Diophantine Equation

$$\boldsymbol{x^2-(a^2b^2+b)y^2-(4c+2)x+4(a^2b^2+b)y-4(a^2b^2+b-c^2-c)=0}$$

Consider the Diophantine Equation

E: $\boldsymbol{x^2-(a^2b^2+b)y^2-(4c+2)x+4(a^2b^2+b)y-4(a^2b^2+b-c^2-c)=0}$ (1)

to be solved over $\boldsymbol{Z}$. It is no longer effortless to remedy and locate the nature and homes of the options of (1). So we apply a linear transformation T to (1) to switch to a less complicated structure for which we can decide the integral solutions.

Let $T=\begin{cases}x=u+h\\ y=v+k\end{cases}$ be the transformation for some $h,k\in Z$. In this case h,k is called the base of T and is denoted by $T[h;k]=\{h,k\}$. Applying T to E, we get

$$\boldsymbol{T(E)=\bar{E}: (u+h)^2-(a^2b^2+b)(v+k)^2-(4c+2)(u+h)+4(a^2b^2+b)(v+k)-4(a^2b^2+b-c^2-c)=0}$$

equating the coefficients of u and v to zero, we get $h = 2c + 1$ and $k = 2$. Hence for $x = u + 2c + 1$ and $y = v + 2$, we have the Diophantine equation

$$\bar{P}: u^2 - (a^2b^2 + b)v^2 = 1 \qquad (2)$$

which is a Pell equation.

The following Corollaries are used to conclude our main theorems.

Corollary 2.1 Let $x_1 + y_1\sqrt{D}$ be the fundamental solution to the equation $x^2 - Dy^2 = 1$. Then all positive integer solutions to the equation $x^2 - Cy^2 = 1$ are given by

$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n$$

with $n \geq 1$.

Corollary 2.3 Let $D = a^2b^2 + b$. Then the continued fraction expansion of $\sqrt{D}$, are given by

$$\sqrt{D} = \begin{cases} [ab, \overline{2a, 2ab}] & if\ t > 1 \\ [a. \overline{2a}] & if\ t = 1 \end{cases}$$

Corollary 2.4 Let $D = a^2b^2 + b$. Then the fundamental solution of the equation $x^2 - Dy^2 = 1$ is given by

$$x_1 + y_1\sqrt{D} = (2a^2b + 1) + 2a\sqrt{D}$$

Proof.

The period length of the continued fraction expansion of $\sqrt{a^2b^2 + b}$ is 2, by Corollary 2.3. therefore, the fundamental solution of the equation $x^2 - Dy^2 = 1$ is $p_1 + q_1\sqrt{C}$. Since $\frac{p_1}{q_1} = ab + \frac{1}{2a} = \frac{2a^2b+1}{2a}$, the proof follows.

1. Main Theorems

Theorem 3.1 Let $a, b > 0$ and $D = a^2b^2 + b$. Then all positive solutions of the equation

$u^2 - Dv^2 = 1$ are given by

$$(u, v) = \left(\frac{2^{2n-1}}{b^n}(ab)^{\lfloor n \rfloor} q_{2n}\left(a, b, \frac{1}{4a}, 1\right), \frac{2^{2n-1}}{ab^n}(ab)^{\lfloor n \rfloor} f_{2n}\left(a, b, \frac{1}{4a}\right)\right)$$

with $n \geq 1$.

Proof.

By Corollary 2.1, Corollary 2.2, and 2.4, all positive integer solutions of the equation $u^2 - Cv^2 = 1$ are given by

$$u_n + v_n\sqrt{D} = \left((2a^2b + 1) + 2a\sqrt{D}\right)^n$$

with $n \geq 1$. Let $\alpha_1 = (2a^2b + 1) + 2a\sqrt{D}$ and $\beta_1 = (2a^2b + 1) - 2a\sqrt{D}$. Then,

$$u_n + v_n\sqrt{D} = {\alpha_1}^n \text{ and } u_n - v_n\sqrt{D} = {\beta_1}^n$$

Thus, it follows that

$$u_n = \frac{{\alpha_1}^n + {\beta_1}^n}{2} \text{ and } v_n = \frac{{\alpha_1}^n - {\beta_1}^n}{2\sqrt{D}}$$

Let

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2} \text{ and } \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}$$

Take $c = \frac{1}{4a}$, we get

$$\alpha = \frac{ab + \sqrt{a^2b^2 + b}}{2} \text{ and } \beta = \frac{ab - \sqrt{a^2b^2 + b}}{2}$$

Thus, $4\alpha^2 = b\left[(2a^2b + 1) + 2a\sqrt{a^2b^2 + b}\right] = \alpha_1$ and $4\beta^2 = b\left[(2a^2b + 1) - 2a\sqrt{a^2b^2 + b}\right] = \beta_1$

Therefore, we get,

$$u_n = \frac{\left(\frac{4\alpha^2}{b}\right)^n + \left(\frac{4\beta^2}{b}\right)^n}{2} = \frac{2^{2n-1}}{b^n}(\alpha^{2n} + \beta^{2n})$$

$$= \frac{2^{2n-1}}{b^n}(ab)^{\lfloor n \rfloor} q_{2n}\left(a, b, \frac{1}{4a}, 1\right) \quad \text{by (2)}$$

and

$$v_n = \frac{\left(\frac{4\alpha^2}{b}\right)^n - \left(\frac{4\beta^2}{b}\right)^n}{2\sqrt{a^2b^2 + b}} = \frac{2^{2n-1}}{b^n}\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}$$

$$= \frac{2^{2n-1}}{ab^n}(ab)^{\lfloor n \rfloor} f_{2n}\left(a, b, \frac{1}{4a}\right) \quad \text{by (1)}$$

Thus, $(u, v) = \left(2^{2n-1}m^{\lfloor n \rfloor} q_{2n}\left(1, m, \frac{1}{4m}, 1\right), 2^{2n-1}m^{\lfloor n \rfloor} f_{2n}\left(1, m, \frac{1}{4m}\right)\right)$

Corollary 3.1. The base of the transformation $T = \begin{cases} x = u + h \\ y = v + k \end{cases}$ is the fundamental solution of (2) that is, $T[h; k] = \{h, k\} = \{u1, v1\}$.

Proof.

We proved that $(u_1, v_1) = (2a^2b + 1, 2a)$ is the fundamental solution of (2). Also we showed that $h = 2c + 1$ and $k = 2$. Hence the result is clear.

We see that (1) could be transformed into (2) using the transformation $T = \begin{cases} x = u + h \\ y = v + k \end{cases}$

Also we showed that $x = u + 2c + 1$ and $y = v + 2$. So we can retransfer all results

from (2) to (1) by using the inverse of the transformation $T = \begin{cases} x = u + h \\ y = v + k \end{cases}$ we can give the following main theorem.

Theorem 3.2. Let (1) be the Diophantine equation. Then we give the following:

1. The fundamental solution of (2) is $(x_1, y_1) = (2a^2b + 2c + 2, 2a + 2)$.

2. For the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n + 2c + 1, v_n + 2)\}$ where $\{u_n, v_n\}$ is the solution of (2). (x_n, y_n) is a solution of (1). So (1) has infinitely many integer solutions.

III. Conclusion

Diophantine equations are prosperous in variety. There is no ordinary approach to for discovering all viable options for Diophantine equations. The approach appears to be easy however it is very hard for attaining the solutions.

References

[1]. Dickson,L.E., History of Theory of Numbers, Vol.II, Chelsea Publishing Company, New York (1952).

[2]. Smith, D.E., History of Mathematics, Vol.I and II, Dover Publications, New York (1953).

[3]. Titu Andreescu, Dorin Andrica and Ion Cucurezeanu., An introduction to Diophantine Equations, Birhauser, New York, 2010.

[4]. Yayenie, O. A note on generalized Fibonacci sequences. Appl. Math. Comput. 2011, 217, 5603-5611.

[5]. Choo, Y. On the generalized bi-periodic Lucas quaternions. Int. J. Math. Anal.2020, 14, 137-145.

[6]. Bilgici, G. Two generalizations of Lucas sequence. Appl. Math. Comput. 2014, 245, 526-538.

[7]. Choo, Y. Relations between Generalized Bi-Periodic Fibonacci and Lucas Sequences, MDPI, 2020, 8, 1527.

[8]. Keskin, R. and Duman, M.G. Positive integer solutions of some Pell equations, Palestine Journal of Mathematics, Vol. 8(2) (2019), 213-226.

[9]. Duman, M.G. Positive Integer Solutions of Some Pell equations, MATHEMATIKA, 2014, Vol. 30(1), 97-108.

[10]. Sriram, S. and Veeramallan, P. Positive Integer Solutions of Some Pell Equations Via Generalized Bi-Periodic Fibonacci and Generalized Bi-Periodic Lucas Sequences, International Journal of Science and Research (IJSR), Vol. 11, Issue 8, August 2022, pp. 1-4.