Solution Of the Diophantine Equation $143^x + 85^y = z^2$

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Abstract: - In this paper, authors studied the Diophantine equation $143^x + 85^y = z^2$, where x, y, z are non-negative integers. Authors proved that $(x, y, z) = (1, 0, 12)$ is the unique non-negative integer solution of this Diophantine equation.

Key Words: ***Catalan's Conjecture, Equation, Solution.***

I. INTRODUCTION

Various problems of Astronomy, Algebra and Trigonometry can be solved by representing them in terms of Diophantine equations [12]. The Diophantine equation $223^x + 241^y = z^2$ was studied by Aggarwal et al. [1]. Aggarwal et al. [2] examined the Diophantine equation $181^x + 199^y = z^2$ and proved that this equation has no solution in non-negative integers. Aggarwal and Sharma [3] analyzed the non-linear Diophantine equation $379^x + 397^y = z^2$ for non-negative integer solution. The Diophantine equation $193^x + 211^y = z^2$ was studied by Aggarwal [4]. Aggarwal and Kumar [5] examined the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$. Aggarwal and Upadhyaya [6] studied the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$ and proved that this Diophantine equation has a unique solution in non-negative integers. Gupta et al. [7] examined the Diophantine equation ${M_5}^p + {M_7}^q = r^2$. Bhatnagar and Aggarwal [8] studied the Diophantine equation $421^p + 439^q = r^2$ and proved that this equation has no solution in non-negative integers.

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Gupta et al. [9] studied the non-linear exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = z^2$. Gupta et al. [10] examined non-linear exponential Diophantine equation $x^\alpha + (1 + my)^\beta = z^2$. Hoque and Kalita [11] determined the solution of the Diophantine equation $(p^q - 1)^x + p^{qy} = z^2$. Kumar et al. [13] proved that the Diophantine equation $601^p + 619^q = r^2$ has no solution in the set of non-negative integers. Kumar et al. [14] determined that the Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ has no non-negative integer solution. Kumar et al. [15] proved that the Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ is not solvable in the set of non-negative integers. The Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ was studied by Mishra et al. [16]. Sroysang [18-22] examined the Diophantine equations $323^x + 325^y = z^2$, $3^x + 45^y = z^2$, $143^x + 145^y = z^2$, $3^x + 85^y = z^2$ and $4^x + 10^y = z^2$ for non-negative integer solution.

The main aim of this paper is to study the Diophantine equation $143^x + 85^y = z^2$, where x, y, z are non-negative integers, for non-negative integer solution.

II. PRELIMINARIES

PROPOSITION 2.1 Catalan's Conjecture [17]: The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$, has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

LEMMA 2.2 The Diophantine equation $143^x + 1 = z^2$, where x, z are non-negative integers, has a unique solution $(x, z) = (1,12)$.

PROOF: Suppose that x, z are non-negative integers such that $143^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Now $z^2 = 143^x + 1 \geq 143^1 + 1 = 144$. Thus $z \geq 12$. Now, we consider the equation $z^2 - 143^x = 1$. By Proposition 2.1, we have $x = 1$. It follows that $z^2 = 144$. Hence, $z = 12$.

LEMMA 2.3 The Diophantine equation $85^y + 1 = z^2$, where y, z are non-negative integers, has no non-negative integer solution.

PROOF: Suppose that y, z are non-negative integers such that $85^y + 1 = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Now $z^2 = 85^y + 1 \geq 85^1 + 1 = 86$. Thus $z \geq 10$. Now, we consider the equation $z^2 - 85^y = 1$. By Proposition 2.1, we have $y = 1$. It follows that $z^2 = 86$. This is a contradiction. Hence, the Diophantine equation $85^y + 1 = z^2$, where y, z are non-negative integers, has no non-negative integer solution.

III. MAIN RESULTS

THEOREM 3.1 $(x, y, z) = (1, 0, 12)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = z^2$, where x, y, z are non-negative integers.

PROOF: Let x, y, z be non-negative integers such that $143^x + 85^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Note that z is even. Then $z^2 \equiv 0 (mod\ 4)$. Since $85^y \equiv 1 (mod\ 4)$, it follows that $143^x \equiv 3 (mod\ 4)$. We obtain that x is odd. Now, we will divide the number y into two cases.

CASE $y = 0$. By Lemma 2.2, we obtain that $x = 1$ and $z = 12$.

CASE $y \geq 1$. Then $85^y \equiv 0 (mod\ 5)$. Note that $143^x \equiv 3 (mod\ 5)$ or $143^x \equiv 2 (mod\ 5)$. Then $z^2 \equiv 2 (mod\ 5)$ or $z^2 \equiv 3 (mod\ 5)$. In fact, $z^2 \equiv 0 (mod\ 5)$ or $z^2 \equiv 1 (mod\ 5)$ or $z^2 \equiv 4 (mod\ 5)$. This is a contradiction.

Hence, $(x, y, z) = (1, 0, 12)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = z^2$, where x, y, z are non-negative integers.

COROLLARY 3.2 The Diophantine equation $143^x + 85^y = w^4$, where x, y, w are non-negative integers, has no non-negative integer solution.

PROOF: Let x, y, w be non-negative integers such that $143^x + 85^y = w^4$. Let $z = w^2$. Then the equation $143^x + 85^y = w^4$ becomes $143^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $w^2 = z = 12$. This is a contradiction. Hence, the Diophantine equation $143^x + 85^y = w^4$, where x, y, w are non-negative integers, has no non-negative integer solution.

COROLLARY 3.3 $(x, y, s) = (1, 0, 6)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = 4s^2$, where x, y, s are non-negative integers.

PROOF: Let x, y, s be non-negative integers such that $143^x + 85^y = 4s^2$. Let $z = 2s$. Then the equation $143^x + 85^y = 4s^2$ becomes $143^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $2s = z = 12$. Thus $s = 6$. Hence, $(x, y, s) = (1, 0, 6)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = 4s^2$, where x, y, s are non-negative integers.

COROLLARY 3.4 $(x, y, u) = (1, 0, 4)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = 9u^2$, where x, y, u are non-negative integers.

PROOF: Let x, y, u be non-negative integers such that $143^x + 85^y = 9u^2$. Let $z = 3u$. Then the equation $143^x + 85^y = 9u^2$ becomes $143^x + 885^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $3u = z = 12$. Thus $u = 4$. Hence, $(x, y, u) = (1, 0, 4)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = 9u^2$, where x, y, u are non-negative integers.

COROLLARY 3.5 The Diophantine equation $143^x + 85^y = 4v^4$, where x, y, v are non-negative integers, has no non-negative integer solution.

PROOF: Let x, y, v be non-negative integers such that $143^x + 85^y = 4v^4$. Let $z = 2v^2$. Then the equation $143^x + 85^y = 4v^4$ becomes $143^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $2v^2 = z = 12$. Thus $v^2 = 6$. This is a contradiction. Hence, the Diophantine equation $143^x + 85^y = 4v^4$, where x, y, v are non-negative integers, has no non-negative integer solution.

COROLLARY 3.6 $(x, y, t) = (1, 0, 2)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = 9t^4$, where x, y, t are non-negative integers.

PROOF: Let x, y, t be non-negative integers such that $143^x + 85^y = 9t^4$. Let $z = 3t^2$. Then the equation $143^x + 85^y = 9t^4$ becomes $43^x + 85^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 12)$. Then $3t^2 = z = 12$. Thus $t = 2$. Hence, $(x, y, t) = (1, 0, 2)$ is the unique non-negative integer solution of the Diophantine equation $143^x + 85^y = 9t^4$, where x, y, t are non-negative integers.

IV. CONCLUSION

In this paper, authors successfully studied the Diophantine equation $143^x + 85^y = z^2$, where x, y, z are non-negative integers, and proved that $(x, y, z) = (1, 0, 12)$ is the unique non-negative integer solution of this Diophantine equation with the help of Catalan's Conjecture. The scheme discussed in this paper can be apply in future to solve other Diophantine equations.

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